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GREEN'S FUNCTION FOR GENERAL DISK-CRACK PROBLEMS

YONG LI XU
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

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Abstract—The two-dimensional elasticity problem of a circular disk with an embedded edge dislocation is considered. Using Muskhelishvili's complex variable method and reducing the case under consideration to a Hilbert problem, closed-form solutions are obtained. The dislocation solution may be used as a Green's function to tackle general disk-crack cases. As an example, a disk containing a slant crack subjected to point loads is studied, numerical procedures for calculating the stress intensity factors for both internal and edge cracks are presented.

1. INTRODUCTION

The two-dimensional elasticity case of a circular disk with an embedded edge dislocation is considered. The obtained dislocation solution can be used as a Green's function to tackle general disk-crack cases by virtue of the dislocation pile-up and singular integral equation techniques [e.g. see Erdogan *et al.* (1974) and Xu and Delale (1992)].

On this subject, Delale and Xu (1993) derived a similar solution using Mitchell's general stress functions and the Fourier series technique. Earlier, Dundurs and Sendeckyj (1965) considered an edge dislocation inside a circular inclusion in an infinite matrix. For the special case, i.e. if we let the stiffness of the matrix be zero, Dundurs and Sendeckyj's solution degenerates to that of a circular disk containing an edge dislocation. In both of the above-mentioned solutions, however, the dislocation is located on the horizontal diameter (x-axis). In Delale and Xu's study, they further applied the dislocation solution to disk-crack cases. Due to the limit of the dislocation solutions, the crack is confined to a straight one and has to lie on the horizontal diameter (x-axis).

In this paper, the author considers an edge dislocation embedded in a disk at an arbitrary location. Therefore, the obtained dislocation solution can be used to tackle disk-crack cases with more complicated crack geometry and arbitrary orientations.

Although the solution of the case to be considered can also be obtained by using coordinate transformation of Delale and Xu's† or Dundurs and Sendeckyj's results, as a different approach, Muskhelishvili's complex variable method is used. Further, a disk containing a slant crack subjected to point forces is considered to illustrate the application of the dislocation solution. The numerical procedures for calculating the stress intensity factors for both internal and edge cracks are presented.

2. DISLOCATION SOLUTION

The geometry of the dislocation case, as shown in Fig. 1, is a circular disk ($r \leqslant R$) containing an edge dislocation with Burgers vector (b_x, b_y, 0) at $z_0 = \rho\,e^{i\alpha}$. The disk is free from traction at $r = R$, the boundary condition may be written as,

† Only the solution for Burgers vector, $b_i = [0, b_y, 0]$ is obtained. In order to use the transformation method, the solution for $b_i = [b_x, 0, 0]$ is also needed.

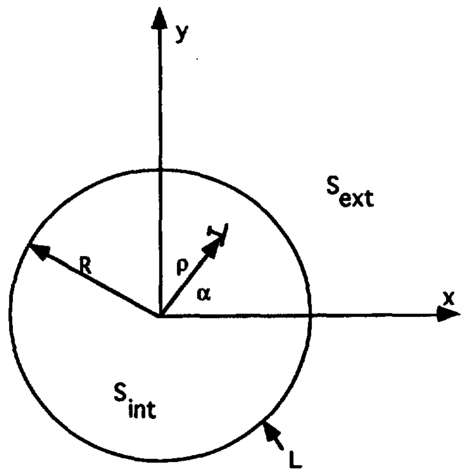

Fig. 1. A circular disk with an edge dislocation embedded at an arbitrary location.

$$\sigma_{rr}+\mathrm{i}\tau_{r\theta} = 0, \quad r = R, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{1}$$

In Muskhelishvili's complex variable technique, the stresses and displacements for a two-dimensional elasticity case may be expressed in terms of the Kolosov potentials, $\Phi(z)$ and $\Psi(z)$, as follows,

$$\sigma_{rr}+\sigma_{\theta\theta} = 4\,\mathrm{Re}\,\{\Phi(z)\} \tag{2}$$

$$\sigma_{rr}-\sigma_{\theta\theta}+2\mathrm{i}\tau_{r\theta} = 2\frac{z}{\bar{z}}[\bar{z}\Phi'(z)+\Psi(z)] \tag{3}$$

$$2\mu(u_r+u_\theta) = \mathrm{e}^{\mathrm{i}\theta}[\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}] \tag{4}$$

with

$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z) \tag{5}$$

where the prime denotes derivative with respect to z, the overhead bar represents complex conjugate and Re{ } denotes taking the real part of the expression in the parentheses. $\kappa = 3-4v$ for plane strain and $\kappa = (3-v)/(1+v)$ for generalized plane stress, with v being the Poisson's ratio and μ the shear modulus.

Combining (2) and (3) and then taking the complex conjugate yields

$$\sigma_{rr}+\mathrm{i}\tau_{r\theta} = \Phi(z)+\overline{\Phi(z)}-\frac{\bar{z}}{z}[z\overline{\Phi'(z)}+\overline{\Psi(z)}]. \tag{6}$$

Due to the existence of the edge dislocation in the disk, the complex potentials are constructed in the following form,

$$\Phi(z) = \Phi_0(z)+\Phi_1(z) \tag{7}$$

$$\Psi(z) = \Psi_0(z)+\Psi_1(z) \tag{8}$$

where $\Phi_0(z)$ and $\Psi_0(z)$ are the stress functions for an edge dislocation embedded in an infinite plane. $\Phi_1(z)$ and $\Psi_1(z)$ are the nonsingular part of the potentials, and they are determined in such a way that the boundary conditions can be satisfied.

The solutions for an edge dislocation with Burgers vector $(b_x, b_y, 0)$ at the point $z_0 = \rho\mathrm{e}^{\mathrm{i}\alpha}$ in an infinite plane are known as (Muskhelishvili, 1953),

$$\Phi_0(z) = \frac{A}{z-z_0} \tag{9}$$

$$\Psi_0(z) = \frac{\bar{A}}{z-z_0} + \frac{A\overline{z_0}}{(z-z_0)^2} \tag{10}$$

and

$$A = \frac{\mu(b_y - \mathrm{i}b_x)}{\pi(1+\kappa)}. \tag{11}$$

From eqns (6), (7), (8), (9) and (10), we may obtain

$$\Phi_1(t) + \overline{\Phi_1(t)} - \frac{R^2}{t}\overline{\Phi_1'(t)} - \frac{R^2}{t^2}\overline{\Psi_1(t)} = g(t), \quad t = R\,\mathrm{e}^{\mathrm{i}\theta}, \quad 0 \leqslant \theta \leqslant 2\pi \tag{12}$$

where

$$g(t) = -\frac{A}{t-z_0} - \frac{\bar{A}t}{R^2-\bar{z}_0 t} - \frac{\bar{A}R^2 t}{(R^2-\bar{z}_0 t)^2} + \frac{AR^2}{t(R^2-\bar{z}_0 t)} + \frac{\bar{A}R^2 z_0}{(R^2-\bar{z}_0 t)^2}. \tag{13}$$

In the usual manner (Milne-Thomson, 1960), we may extend $\Phi_1(z)$ to the region outside the disk by defining

$$\Phi_1\left(\frac{R^2}{\bar{z}}\right) = -\overline{\Phi_1(z)} + \bar{z}\overline{\Phi_1'(z)} + \frac{\bar{z}^2}{R^2}\overline{\Psi_1(z)}, \quad (z \in S_{\mathrm{int}}) \tag{14}$$

or equivalently,

$$\Phi_1(z) = -\bar{\Phi}_1\left(\frac{R^2}{z}\right) + \frac{R^2}{z}\bar{\Phi}_1'\left(\frac{R^2}{z}\right) + \frac{R^2}{z^2}\bar{\Psi}_1\left(\frac{R^2}{z}\right)^{\dagger}, \quad (z \in S_{\mathrm{ext}}). \tag{15}$$

From eqn (14), we may also write

$$\Psi_1(z) = \frac{R^2}{z^2}\Phi_1(z) - \frac{R^2}{z}\Phi_1'(z) + \frac{R^2}{z^2}\bar{\Phi}_1\left(\frac{R^2}{z}\right), \quad (z \in S_{\mathrm{int}}). \tag{16}$$

Since $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in the disk domain, from the definition in eqn (15), $\Phi_1(z)$ should also be holomorphic outside the disk, including at infinity, where its principal part is a complex constant.

Combining eqns (6) and (16), and imposing the boundary condition (12) it follows,

$$\Phi_1^+(t) - \Phi_1^-(t) = g(t), \quad t = R\,\mathrm{e}^{\mathrm{i}\theta}, \quad 0 \leqslant \theta \leqslant 2\pi \tag{17}$$

where "+" and "−" represent approaching the boundary $r = R$ from the outside and inside of the circle, respectively. The solution of eqn (17) can be found using the Plemelj formula

† $\overline{f(z)} = \bar{f}(\bar{z})$, to obtain $\bar{f}(R^2/z)$, simply write R^2/z for $\bar{z}$ in the expression of $\bar{f}(\bar{z})$.

$$\Phi_1(z) = \frac{1}{2\pi i}\int_C \frac{g(t)}{t-z}\,dt + c_1. \tag{18}$$

It turns out that

$$\Phi_1(z) = \begin{cases} \dfrac{\bar{A}}{\bar{z}_0} + \dfrac{A\bar{z}_0}{R^2-\bar{z}_0 z} - \dfrac{\bar{A}R^2}{(R^2-\bar{z}_0 z)^2}\left(\dfrac{R^2}{\bar{z}_0} - z_0\right) + c_1, & (z \in S_{\text{int}}) \\ \dfrac{A}{z-z_0} - \dfrac{A}{z} + c_1, & (z \in S_{\text{ext}}) \end{cases} \tag{19}$$

where c_1 is a complex constant, and may be determined by substituting eqn (19) into (16) and using the fact that $\Psi_1(z)$ is holomorphic in the disk domain. By so doing, we may obtain

$$\text{Re}\,\{c_1\} = -\frac{\mu(b_y\rho\cos\alpha - b_x\rho\sin\alpha)}{\pi(1+\kappa)R^2}. \tag{20}$$

Due to the fact that the complex potential $\Phi_1(z)$ can differ by an imaginary number without causing any change of the stresses, we may simply let the imaginary part of c_1 be zero.

It may be noted that we may also determine c_1 by letting $z = 0$ in eqn (14) to obtain

$$\Phi_1(\infty) = -\overline{\Phi_1(0)} \tag{21}$$

then inserting eqn (19) into (21), which gives the same result as in eqn (20).

Substituting eqn (19) to (16) yields $\Psi_1(z)$ as below,

$$\Psi_1(z) = \frac{\bar{A}\bar{z}_0}{R^2-\bar{z}_0 z} + \frac{\bar{A}(R^2-\bar{z}_0 z_0)\bar{z}_0 - A\bar{z}_0^3}{(R^2-\bar{z}_0 z)^2} + \frac{2\bar{A}R^2\bar{z}_0(R^2-z_0\bar{z}_0)}{(R^2-\bar{z}_0 z)^3} \quad (z \in S_{\text{int}}). \tag{22}$$

With the complex potentials given in eqns (9), (10), (19) and (22), the stress fields can be readily obtained from (2) and (3).

The closed-form stress distributions in the dislocation-embedded disk have been found and are given in the Appendix, and the displacement fields may then be obtained by using the constitutive and kinematic equations.

3. DISK-CRACK PROBLEMS

The dislocation solution obtained above may be used as a Green's function to tackle disk-crack cases using dislocation pile-up and singular integral equation techniques. Following the standard procedures (see Fig. 2), i.e. first, by superposition of the crack case

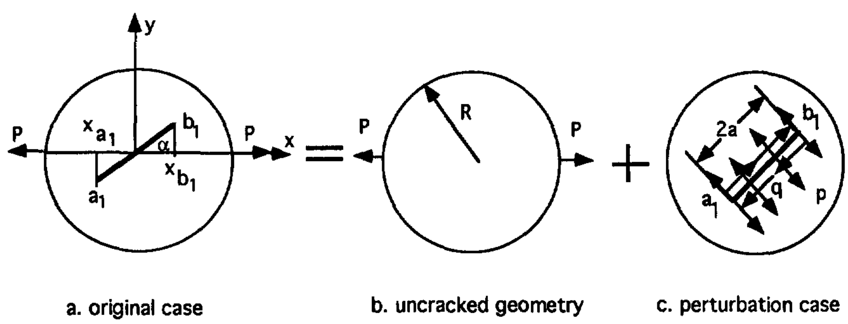

Fig. 2. Superposition with uncracked geometry solution.

under consideration with the uncracked geometry solution, the original crack problem may be reduced to a perturbation case, in which the crack singular essence remains the same but the loading is only the crack surface self-equilibrating pressure and shear stress. Next, by integrating the dislocation solution, a singular integral equation may be generated. The stress intensity factors of the crack are related to the unknown function in the integral equation and thus can be formulated in terms of the solution of the singular integral equation, which can then be solved by a collocation method in a very desirable accuracy.

In what follows, a circular disk (radius R) containing a slant radial crack (from $\rho = a_1$, to $\rho = b_1$, $\theta = \alpha$) subjected to a pair of point forces, P, at $\theta = 0$ and $\theta = \pi$ is considered [see Fig. 2(a)]. Following the procedure, the original case is translated to its corresponding perturbation case, the boundary conditions of which can be written as follows,

$$\sigma_{\theta\theta}(R,\theta) = \tau_{r\theta}(R,\theta) = 0, \quad 0 \leqslant \theta \leqslant 2\pi \tag{23}$$

$$\sigma_{\theta\theta}(\rho,\theta) = p(\rho) \quad \theta = \alpha, \quad a_1 < \rho < b_1 \tag{24}$$

$$\tau_{r\theta}(\rho,\theta) = q(\rho) \quad \theta = \alpha, \quad a_1 < \rho < b_1 \tag{25}$$

where $p(\rho) = -\sigma^*_{\theta\theta}(\rho,\alpha)$ and $q(\rho) = -\tau^*_{r\theta}(\rho,\alpha)$, $\sigma^*_{\theta\theta}$ and $\tau^*_{r\theta}$ are hoop and shear stresses in the uncracked geometry case (the problem with the same load and geometry except without the crack), which are known for the case in discussion [e.g. see Muskhelishvili (1953)],

$$\sigma^*_{\theta\theta}(\rho,\alpha) = \frac{2P}{\pi}\left[\frac{(R-\rho\cos\alpha)^2(\rho\cos\alpha - R\cos 2\alpha)}{(R^2+\rho^2-2R\rho\cos\alpha)^2} - \frac{(R+\rho\cos\alpha)^2(\rho\cos\alpha + R\cos 2\alpha)}{(R^2+\rho^2+2R\rho\cos\alpha)^2}\right.$$
$$\left. + \frac{\cos^2\alpha(R-\rho\cos\alpha)}{R^2+\rho^2-2R\rho\cos\alpha} + \frac{\cos^2\alpha(R+\rho\cos\alpha)}{R^2+\rho^2+2R\rho\cos\alpha}\right] - \frac{P}{\pi R} \tag{26}$$

$$\tau^*_{r\theta}(\rho,\alpha) = \frac{2P}{\pi}\left[\frac{(R-\rho\cos\alpha)^2(\rho\sin 3\alpha - R\sin 2\alpha)}{(R^2+\rho^2-2R\rho\cos\alpha)^2} - \frac{(R+\rho\cos\alpha)^2(\rho\sin 3\alpha + R\sin 2\alpha)}{(R^2+\rho^2+2R\rho\cos\alpha)^2}\right.$$
$$\left. + \frac{\sin 2\alpha(R-\rho\cos\alpha)}{2(R^2+\rho^2-2R\rho\cos\alpha)} + \frac{\sin 2\alpha(R+\rho\cos\alpha)}{2(R^2+\rho^2+2R\rho\cos\alpha)}\right]. \tag{27}$$

If we define

$$f(x) = \frac{\partial[u_y(x,y^+) - u_y(x,y^-)]}{\partial x}, \quad (x_{a_1} < x < x_{b_1}) \tag{28}$$

$$g(x) = \frac{\partial[u_x(x,y^+) - u_x(x,y^-)]}{\partial x}, \quad (x_{a_1} < x < x_{b_1}) \tag{29}$$

where u_x and u_y are the Cartesian coordinate displacements, and x and y are Cartesian coordinates of the point on the crack. y^- and y^- denote the point on the upper and lower edges of the crack, respectively, and x_{a_1} and x_{b_1} are the projections of a_1 and b_1 on the x-axis.

By integrating the dislocation solution and applying the boundary conditions (24) and (25), we may obtain

$$\int_{a_1}^{b_1} \frac{f_1(t)\cos\alpha - g_1(t)\sin\alpha}{t-\rho}\,\mathrm{d}t + \int_{a_1}^{b_1} [f_1(t)\cos\alpha - g_1(t)\sin\alpha]k_1(\rho,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}p(\rho),$$

$$(a_1 < \rho < b_1) \tag{30}$$

and

$$\int_{a_1}^{b_1} \frac{f_1(t)\sin\alpha + g_1(t)\cos\alpha}{t-\rho}\,\mathrm{d}t + \int_{a_1}^{b_1} [f_1(t)\sin\alpha + g_1(t)\cos\alpha]k_2(\rho,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}q(\rho),$$

$$(a_1 < \rho < b_1) \tag{31}$$

with

$$k_1(\rho,t) = \frac{t}{R^2} + \frac{\rho-t}{R^2-\rho t} - \frac{2R^2t - t^3 - R^2\rho}{(R^2-\rho t)^2} + \frac{R^2(R^2-t^2)(\rho-t)}{(R^2-\rho t)^3} \tag{32}$$

$$k_2(\rho,t) = \frac{R^2(R^2-t^2)(\rho-t)}{(R^2-\rho t)^3} - \frac{t}{R^2-\rho t} \tag{33}$$

where

$$f_1(t) = f(t\cos\alpha), \quad g_1(t) = g(t\cos\alpha). \tag{34}$$

a. *Internal crack*

If $-R < a_1 < b_1 < R$, $\theta = \alpha$, which means that no crack end stretches to the edge of the disk, the crack is referred to as an internal radial crack. For such cases, $k_1(\rho, t)$ and $k_2(\rho, t)$ are all Fredholm kernels, and the singularity only comes from the Cauchy kernel terms. For an embedded crack, it follows,

$$\int_{a_1}^{b_1} f_1(t)\,\mathrm{d}t = 0 \tag{35}$$

and

$$\int_{a_1}^{b_1} g_1(t)\,\mathrm{d}t = 0. \tag{36}$$

The singularity of the internal crack is known, and we may write

$$f_1(t) = \frac{F_1(t)}{\sqrt{(t-a_1)(b_1-t)}}, \quad (a_1 < t < b_1) \tag{37}$$

and

$$g_1(t) = \frac{G_1(t)}{\sqrt{(t-a_1)(b_1-t)}}, \quad (a_1 < t < b_1) \tag{38}$$

where $F_1(t)$ and $G_1(t)$ are continuous functions satisfying $F_1(a_1) \neq 0$, $F_1(b_1) \neq 0$, $G_1(a_1) \neq 0$, $G_1(b_1) \neq 0$.

The stress intensity factors of the crack can be calculated according to the conventional definition, i.e.

$$k_1(a_1 = \lim_{\rho \to a_i} \sqrt{2(a_1 - \rho)}\, \sigma_{\theta\theta}(\rho, \alpha)$$

$$= \frac{2\mu \cos\alpha}{1+\kappa} \lim_{\rho \to a_1^+} \sqrt{2(\rho - a_1)}\, [f_1(\rho)\cos\alpha - g_1(\rho)\sin\alpha]$$

$$= \frac{2\mu\cos\alpha}{1+\kappa}\sqrt{2}\frac{[F_1(a_1)\cos\alpha - G_1(a_1)\sin\alpha]}{\sqrt{b_1 - a_1}} \tag{39}$$

and

$$k_1(b_1) = -\frac{2\mu\cos\alpha}{1+\kappa}\sqrt{2}\frac{[F_1(b_1)\cos\alpha - G_1(b_1)\sin\alpha]}{\sqrt{b_1 - a_1}} \tag{40}$$

$$k_2(a_1) = \lim_{\rho \to a_i} \sqrt{2(a_1 - \rho)}\, \tau_{r\theta}(\rho, \alpha)$$

$$= \frac{2\mu \cos\alpha}{1+\kappa} \lim_{\rho \to a_1^+} \sqrt{2(\rho - a_1)}\, [f_1(\rho)\sin\alpha + g_1(\rho)\cos\alpha]$$

$$= \frac{2\mu\cos\alpha}{1+\kappa}\sqrt{2}\frac{[F_1(a_1)\sin\alpha + G_1(a_1)\cos\alpha]}{\sqrt{b_1 - a_1}} \tag{41}$$

$$k_2(b_1) = -\frac{2\mu\cos\alpha}{1+\kappa}\sqrt{2}\frac{[F_1(b_1)\sin\alpha + G_1(b_1)\cos\alpha]}{\sqrt{b_1 - a_1}}. \tag{42}$$

Introducing the following transformations in eqns (30), (31), (35) and (36),

$$\rho = \frac{b_1 - a_1}{2}s + \frac{b_1 + a_1}{2}, \quad \text{when } a_1 < \rho < b_1, \quad -1 < s < 1$$

$$t = \frac{b_1 - a_1}{2}\tau + \frac{b_1 + a_1}{2}, \quad \text{when } a_1 < t < b_1, \ -1 < \tau < 1 \tag{43}$$

and letting

$$f_2(\tau) = f_1(t),\ g_2(\tau) = g_1(t), \quad p_2(s) = p(\rho), \quad q_2(s) = q(\rho)$$
$$K_1(s,\tau) = k_1(\rho, t), \quad K_2(s,\tau) = k_2(\rho, t) \tag{44}$$

we obtain,

$$\int_{-1}^{1} \frac{f_2(\tau)\cos\alpha - g_2(\tau)\sin\alpha}{\tau - s}\,d\tau + \frac{b_1 - a_1}{2}\int_{-1}^{1}[f_2(\tau)\cos\alpha - g_2(\tau)\sin\alpha]K_1(s,\tau)\,d\tau$$
$$= \frac{\pi(1+\kappa)}{2\mu\cos\alpha}p_2(s), \quad (-1 < s < 1) \tag{45}$$

$$\int_{-1}^{1} \frac{f_2(\tau)\sin\alpha + g_2(\tau)\cos\alpha}{\tau - s}\,d\tau + \frac{b_1 - a_1}{2}\int_{-1}^{1}[f_2(\tau)\sin\alpha + g_2(\tau)\cos\alpha]K_2(s,\tau)\,d\tau$$
$$= \frac{\pi(1+\kappa)}{2\mu\cos\alpha}q_2(s), \quad (-1 < s < 1) \tag{46}$$

$$\int_{-1}^{1} f_2(\tau)\,d\tau = 0 \tag{47}$$

$$\int_{-1}^{1} g_2(\tau)\,\mathrm{d}\tau = 0. \tag{48}$$

Accordingly, we may write

$$f_2(\tau) = \frac{F_2(\tau)}{\sqrt{(1-\tau^2)}}, \quad (-1 < \tau < 1) \tag{49}$$

and

$$g_2(\tau) = \frac{G_2(\tau)}{\sqrt{(1-\tau^2)}}, \quad (-1 < \tau < 1). \tag{50}$$

Using the Labatto–Chebyshev integration method [see Erdogan and Gupta (1972), and Ioakimidas and Theocaris (1980)], eqns (45)–(48) can be discretized into two sets of N simultaneous algebraic equations:

set 1:

$$\begin{cases} \sum_{i=1}^{n} \left[\lambda_i \dfrac{[\cos\alpha F_2(\tau_i) - \sin\alpha G_2(\tau_i)]}{\tau_i - s_j} + \lambda_i \dfrac{b_1 - a_1}{2} K_1(s_j, \tau_i)\,[\cos\alpha F_2(\tau_i) - \sin\alpha G_2(\tau_i)] \right] \\ \qquad\qquad = \dfrac{\pi(1+\kappa)}{2\mu\cos\alpha} p_2(s_j), \quad (j = 1, 2, \ldots, n-1) \\ \sum_{i=1}^{n} \{\lambda_i\,[\cos\alpha F_2(\tau_i) - \sin\alpha G_2(\tau_i)]\} = 0; \end{cases} \tag{51}$$

set 2:

$$\begin{cases} \sum_{i=1}^{n} \left[\lambda_i \dfrac{[\sin\alpha F_2(\tau_i) + \cos\alpha G_2(\tau_i)]}{\tau_i - s_j} + \lambda_i \dfrac{b_1 - a_1}{2} K_2(s_j, \tau_i)\,[\sin\alpha F_2(\tau_i) + \cos\alpha G_2(\tau_i)] \right] \\ \qquad\qquad = \dfrac{\pi(1+\kappa)}{2\mu\cos\alpha} q_2(s_j), \quad (j = 1, 2, \ldots, n-1) \\ \sum_{i=1}^{n} \{\lambda_i\,[\sin\alpha F_2(\tau_i) + \cos\alpha G_2(\tau_i)]\} = 0 \end{cases} \tag{52}$$

with weight coefficients λ_is as

$$\lambda_i = \begin{cases} \dfrac{\pi}{2(n-1)}, & \text{when } i = 1 \text{ or } n \\ \dfrac{\pi}{(n-1)}, & \text{when } i = 2, 3, \ldots, n-1 \end{cases} \tag{53}$$

and τ_i and s_j satisfying

$$\begin{aligned} &T_{n-1}(s_j) = 0, \quad j = 1, 2, 3, \ldots, n-1 \\ &U_{n-1}(\tau_i) = 0, \quad i = 1, 2, 3, \ldots, n-2 \text{ and } \tau = \pm 1 \end{aligned} \tag{54}$$

where $T_{n-1}(x)$ and $U_{n-2}(x)$ are Chebyshev polynomials of first and second kinds, respectively.

The stress intensity factors of the crack can be calculated by

$$k_1(a_1) = \frac{2\mu\cos\alpha}{1+\kappa}\sqrt{b_1-a_1}\,\frac{[F_2(-1)\cos\alpha - G_2(-1)\sin\alpha]}{\sqrt{2}} \tag{55}$$

$$k_1(b_1) = -\frac{2\mu\cos\alpha}{1+\kappa}\sqrt{b_1-a_1}\,\frac{[F_2(1)\cos\alpha - G_2(1)\sin\alpha]}{\sqrt{2}} \tag{56}$$

$$k_2(a_1) = \frac{2\mu\cos\alpha}{1+\kappa}\sqrt{b_1-a_1}\,\frac{[F_2(-1)\sin\alpha + G_2(-1)\cos\alpha]}{\sqrt{2}} \tag{57}$$

$$k_2(b_1) = -\frac{2\mu\cos\alpha}{1+\kappa}\sqrt{b_1-a_1}\,\frac{[F_2(1)\sin\alpha + G_2(1)\cos\alpha]}{\sqrt{2}}. \tag{58}$$

b. *Edge crack*

If one of the crack ends extends to the edge of the disk, e.g. if $-R < a_1 < b_1 = R$, the crack becomes an edge crack. Under such circumstances, the displacement single-valuedness conditions (47) and (48) do not hold true. It can be shown [see Xu and Delale (1992)] that the density functions $f_1(t)$ and $g_1(t)$ can be written as

$$f_1(t) = \frac{F_1(t)}{\sqrt{t-a_1}}, \quad (a_1 < t < b_1) \tag{59}$$

$$g_1(t) = \frac{G_1(t)}{\sqrt{t-a_1}}, \quad (a_1 < t < b_1). \tag{60}$$

By introducing the following transformations,

$$\begin{aligned} \rho &= (b_1-a_1)s + b_1, \quad \text{when } a_1 < \rho < b_1, \quad -1 < s < 0 \\ t &= (b_1-a_1)\tau + b_1, \quad \text{when } a_1 < t < b_1, \quad -1 < \tau < 0 \end{aligned} \tag{61}$$

and letting

$$\begin{aligned} f_2(\tau) = f_1(t), \quad g_2(\tau) = g_1(t), \quad p_2(s) = p(\rho), \quad q_2(s) = q(\rho) \\ K_1(s,\tau) = k_1(\rho,t), \quad K_2(s,\tau) = k_2(\rho,t) \end{aligned} \tag{62}$$

eqns (30), (31), (59) and (60) become

$$\left\{\begin{aligned} &\int_{-1}^{0} \frac{f_2(\tau)\cos\alpha - g_2(\tau)\sin\alpha}{\tau - s}\,d\tau + (b_1-a_1)\int_{-1}^{0} [f_2(\tau)\cos\alpha - g_2(\tau)\sin\alpha]K_1(s,\tau)\,d\tau \\ &\qquad = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}p_2(s), \quad (-1 < s < 0) \\ &\int_{-1}^{0} \frac{f_2(\tau)\sin\alpha + g_2(\tau)\cos\alpha}{\tau - s}\,d\tau + (b_1-a_1)\int_{-1}^{0} [f_2(\tau)\sin\alpha + g_2(\tau)\cos\alpha]K_2(s,\tau)\,d\tau \\ &\qquad = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}q_2(s), \quad (-1 < s < 0) \end{aligned}\right. \tag{63}$$

$$f_2(\tau) = \frac{F_2(\tau)}{\sqrt{1+\tau}}, \quad (-1 < \tau < 0) \tag{64}$$

and

$$g_2(\tau) = \frac{G_2(\tau)}{\sqrt{1+\tau}}, \quad (-1 < \tau < 0). \tag{65}$$

Referring to Gupta and Erdogan (1974), we may extend the definition of $f_2(t)$ and $g_2(t)$ *to* [0,1) as an even continuation, as follows,

$$f_2(\tau) = \frac{F(\tau)}{\sqrt{(1-\tau^2)}}, \quad F_2(\tau) = \frac{F(\tau)}{\sqrt{1-\tau}}, \quad \text{with}\, F(\tau) = F(-\tau), (-1 < \tau < 1) \tag{66}$$

$$g_2(\tau) = \frac{G(\tau)}{\sqrt{(1-\tau^2)}}, \quad G_2(\tau) = \frac{G(\tau)}{\sqrt{1-\tau}}, \quad \text{with}\, G(\tau) = G(-\tau), (-1 < \tau < 1). \tag{67}$$

Discretizing eqns (62) and (63) in a similar manner yields

$$\sum_{i=n+2}^{2n+1} \left[\lambda_i \frac{[\cos\alpha F(\tau_i) - \sin\alpha G(\tau_i)]}{\tau_i - s_j} + \lambda_i (b_1 - a_1) K_1(s_j, \tau_i)[\cos\alpha F(\tau_i) - \sin\alpha G(\tau_i)] \right]$$
$$= \frac{\pi(1+\kappa)}{2\mu\cos\alpha} p_2(s_j) \quad (j = n+2, \ldots, 2n+1) \tag{68}$$

$$\sum_{i=n+2}^{2n+1} \left[\lambda_i \frac{[\sin\alpha F(\tau_i) + \cos\alpha G(\tau_i)]}{\tau_i - s_j} + \lambda_i (b_1 - a_1) K_2(s_j, \tau_i)[\sin\alpha F(\tau_i) + \cos\alpha G(\tau_i)] \right]$$
$$= \frac{\pi(1+\kappa)}{2\mu\cos\alpha} q_2(s_j) \quad (j = n+2, \ldots, 2n+1) \tag{69}$$

where

$$\lambda_i = \frac{\pi}{2n+1}, \quad T_{2n+1}(s_j) = 0, \quad U_{2n}(\tau_i) = 0. \tag{70}$$

The stress intensity factors at a_1 can be calculated by

$$k_1(a_1) = \lim_{\rho \to a_1^-} \sqrt{2(a_1 - \rho)}\, \sigma_{\theta\theta}(\rho, \alpha)$$
$$= \frac{2\mu\cos\alpha}{1+\kappa} \sqrt{(b_1 - a_1)}\, [\cos\alpha F(-1) - \sin\alpha G(-1)] \tag{71}$$

and

$$k_2(a_1) = \lim_{\rho \to a_1^-} \sqrt{2(a_1 - \rho)}\, \tau_{r\theta}(\rho, \alpha)$$
$$= \frac{2\mu\cos\alpha}{1+\kappa} \sqrt{(b_1 - a_1)}\, [\sin\alpha F(-1) + \cos\alpha G(-1)]. \tag{72}$$

4. RESULTS AND DISCUSSIONS

Normalized stress intensity factors of a slant internal disk-crack subjected to uniform crack surface pressure q and shear stress τ_0 for different crack lengths are calculated and

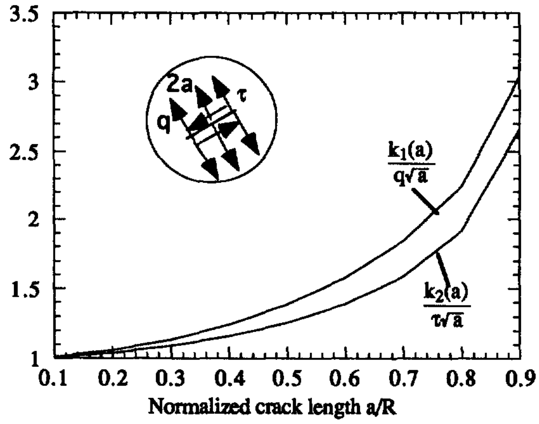

Fig. 3. Stress intensity factors of an internal radial crack in a circular disk subjected to uniform crack surface pressure and shear stress.

given in Fig. 3. For this case, Bowie *et al.* (1970) and Tweed *et al.* (1972) have found mode I stress intensity factors. It may be seen in Fig. 3 that, for the comparable part, the present results are almost exactly the same as theirs. Normalized stress intensity factors for slant internal cracks ($b_1 = -a_1 = a$, central crack) and slant edge cracks ($b_1 = R$, $b_1 - a_1 = 1$) under point forces for various crack lengths and different inclination angles α are computed and presented in Figs 4–7. For $\alpha = 0°$ and $\alpha = 90°$, the present results match those of Xu and Delale (1992) exactly.

Finally, it may be noted that for a linear crack in the isotropic material, such as the example discussed in this work, there is no mode coupling, i.e. k_1 is only due to the crack surface pressure, and k_2 is only induced by the shear stress. In fact, we are able to decompose (or decouple) the integral equations (30) and (31) by defining

$$X_1(t) = f_1(t)\cos\alpha - g_1(t)\sin\alpha \tag{73}$$

$$Y_1(t) = f_1(t)\sin\alpha + g_1(t)\cos\alpha \tag{74}$$

with (35) and (36), we may then obtain

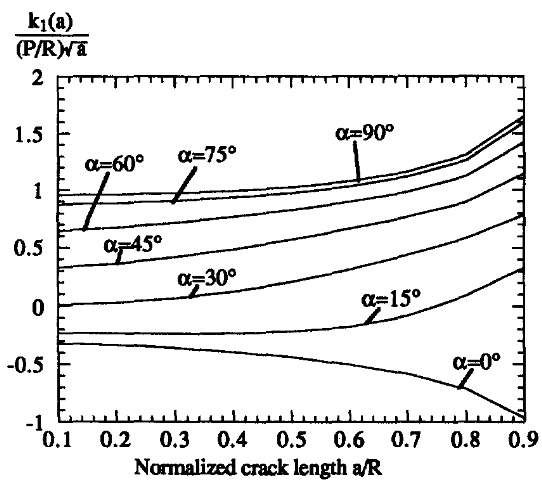

Fig. 4. Stress intensity factors (mode I) of a slant internal crack in a circular disk subjected to point forces.

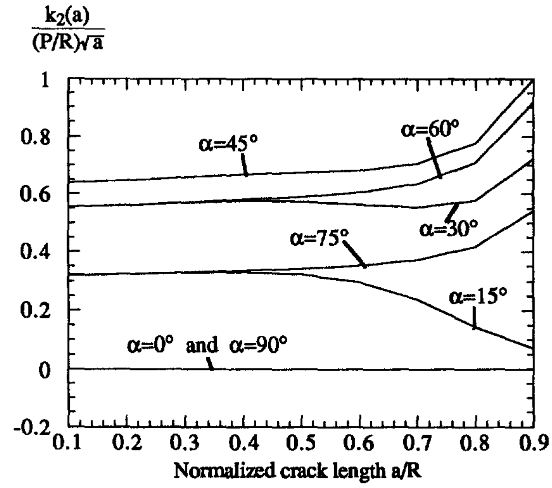

Fig. 5. Stress intensity factors (mode II) of a slant internal crack in a circular disk subjected to point forces.

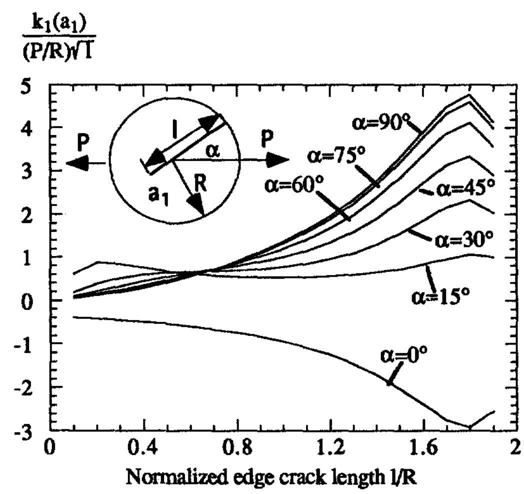

Fig. 6. Stress intensity factors (mode I) of a slant edge crack in circular disk subjected to point forces.

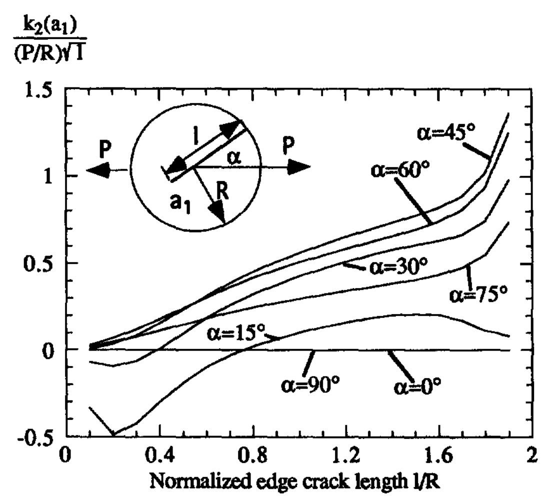

Fig. 7. Stress intensity factors (mode II) of a slant edge crack in a circular disk subjected to point forces.

$$\begin{cases} \int_{a_1}^{b_1} \frac{X_1(t)}{t-\rho}\,\mathrm{d}t + \int_{a_1}^{b_1} X_1(t)k_1(\rho,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}p(\rho), \quad (a_1 < \rho < b_1) \\ \int_{a_1}^{b_1} X_1(t)\,\mathrm{d}t = 0 \end{cases} \tag{75}$$

$$\begin{cases} \int_{a_1}^{b_1} \frac{Y_1(t)}{t-\rho}\,\mathrm{d}t + \int_{a_1}^{b_1} Y_1(t)k_2(\rho,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{2\mu\cos\alpha}q(\rho), \quad (a_1 < \rho < b_1) \\ \int_{a_1}^{b_1} Y_1(t)\,\mathrm{d}t = 0. \end{cases} \tag{76}$$

The equations (75) and (76) are not related and are completely independent. It can be readily shown that $X_1(t)$ and $Y_1(t)$ are nothing but the dislocation density functions in the directions tangential and normal to the crack, respectively.

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APPENDIX

The stress fields of a circular disk of radius, R, containing an edge dislocation at $r = \rho$, $\theta = \alpha$, with Burgers vector $(b_x, b_y, 0)$ are found as below,
$b_x \neq 0$, $b_y = 0$,

$$\begin{aligned} \sigma_{rr}(r,\theta) = \frac{\mu b_x}{\pi(1+\kappa)} \Bigg\{ & \frac{2\rho\sin\alpha - r\sin\theta + \rho\sin(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{-2\rho R^2\sin\alpha + r\rho^2\sin\theta - \rho R^2\sin(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} \\ & - \frac{3r\rho^2\sin(2\alpha-\theta) - 3\rho r^2\sin\alpha - \rho^3\sin(3\alpha-2\theta) + r^3\sin\theta}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} + \frac{2\rho\sin\alpha}{R^2} \\ & - \frac{R^4\rho(R^2-\rho^2-2r^2)\sin(\alpha-2\theta) - R^2r\rho^2(4r^2+5R^2+2\rho^2)\sin(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\ & - \frac{R^2\rho^3(2r^2+R^2)\sin(3\alpha-2\theta) + \rho(2R^6+2r^4\rho^2+8r^2R^4+r^2\rho^2R^2+2r^2\rho^4)\sin\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\ & + \frac{r(4R^6+4\rho^2r^2R^2+\rho^4r^2-2\rho^2R^4+2\rho^4R^2)\sin\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\ & - \frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta) - r^4\rho^3\sin(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \\ & - \frac{2R^2(R^2-\rho^2)[R^4r(3\rho^2+R^2)\sin\theta - 3R^2\rho r^2(R^2+\rho^2)\sin\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \Bigg\} \end{aligned} \tag{A.1}$$

$$
\begin{aligned}
\sigma_{\theta\theta}(r,\theta) = \frac{\mu b_x}{\pi(1+\kappa)} \Bigg\{ & \frac{2\rho\sin\alpha - 3r\sin\theta - \rho\sin(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{-2\rho R^2\sin\alpha + 3r\rho^2\sin\theta + \rho R^2\sin(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} \\
& + \frac{3r\rho^2\sin(2\alpha-\theta) - 3\rho r^2\sin\alpha - \rho^3\sin(3\alpha-2\theta) + r^3\sin\theta}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} + \frac{2\rho\sin\alpha}{R^2} \\
& + \frac{R^4\rho(R^2-\rho^2+2r^2)\sin(\alpha-2\theta) + R^2r\rho^2(4r^2+3R^2-2\rho^2)\sin(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& + \frac{R^2\rho^3(R^2-2r^2)\sin(3\alpha-2\theta) - \rho(2R^6+2r^4\rho^2+8r^2R^4-r^2\rho^2R^2-2r^2\rho^4)\sin\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& + \frac{r(4R^6+4\rho^2r^2R^2-\rho^4r^2+2\rho^2R^4-2\rho^4R^2)\sin\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& + \frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta) - r^4\rho^3\sin(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \\
& + \frac{2R^2(R^2-\rho^2)[rR^4(3\rho^2+R^2)\sin\theta - 3R^2\rho r^2(R^2+\rho^2)\sin\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \Bigg\}
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
\tau_{r\theta}(r,\theta) = \frac{\mu b_x}{\pi(1+\kappa)} \Bigg\{ & \frac{r\cos\theta - \rho\cos(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{\rho R^2\cos(\alpha-2\theta) - r\rho^2\cos\theta}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} \\
& + \frac{r^3\cos\theta + 3r\rho^2\cos(2\alpha-\theta) - 3\rho r^2\cos\alpha - \rho^3\cos(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} \\
& - \frac{R^4\rho(\rho^2-R^2)\cos(\alpha-2\theta) + R^2r\rho^2(R^2+2\rho^2)\cos(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& - \frac{-\rho^3R^4\cos(3\alpha-2\theta) + r\rho^2(r^2\rho^2+2R^4-2R^2\rho^2)\cos\theta - 3r^2R^2\rho^3\cos\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& - \frac{2R^2(R^2-\rho^2)[-\rho R^6\cos(\alpha-2\theta) - r^4\rho^3\cos(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \\
& - \frac{2R^2(R^2-\rho^2)[R^4r(3\rho^2+R^2)\cos\theta - 3R^2\rho r^2(R^2+\rho^2)\cos\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \Bigg\}
\end{aligned} \tag{A.3}
$$

$b_x = 0,\ b_y \neq 0,$

$$
\begin{aligned}
\sigma_{rr}(r,\theta) = \frac{\mu b_y}{\pi(1+\kappa)} \Bigg\{ & \frac{r\cos\theta - 2\rho\cos\alpha + \rho\cos(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} - \frac{3\rho r^2\cos\alpha - r^3\cos\theta - 3r\rho^2\cos(2\alpha-\theta) + \rho^3\cos(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} \\
& + \frac{2\rho R^2\cos\alpha - \rho^2r\cos\theta - \rho R^2\cos(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} - \frac{2\rho\cos\alpha}{R^2} \\
& + \frac{\rho R^4(\rho^2-R^2+2r^2)\cos(\alpha-2\theta) - R^2r\rho^2(4r^2+5R^2+2\rho^2)\cos(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& + \frac{R^2\rho^3(2r^2+R^2)\cos(3\alpha-2\theta) + \rho(2R^6+2r^4\rho^2+8r^2R^4+r^2\rho^2R^2+2r^2\rho^4)\cos\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& - \frac{r(4R^6+4\rho^2r^2R^2+\rho^4r^2-2\rho^2R^4+2\rho^4R^2)\cos\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2} \\
& - \frac{2R^2(R^2-\rho^2)[\rho R^6\cos(\alpha-2\theta) + r^4\rho^3\cos(3\alpha-2\theta) - r^3\rho^2(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \\
& - \frac{2R^4r(R^2-\rho^2)[3\rho r(R^2+\rho^2)\cos\alpha - R^2(3\rho^2+R^2)\cos\theta]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}
\end{aligned} \tag{A.4}
$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\mu b_y}{\pi(1+\kappa)} \left\{ \frac{3r\cos\theta - 2\rho\cos\alpha - \rho\cos(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{2\rho R^2\cos\alpha - 3r\rho^2\cos\theta + \rho R^2\cos(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} \right.$$

$$+ \frac{-3\rho^2 r\cos(2\alpha-\theta) + 3\rho r^2\cos\alpha + \rho^3\cos(3\alpha-2\theta) - r^3\cos\theta}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} - \frac{2\rho\cos\alpha}{R^2}$$

$$+ \frac{R^4\rho(R^2-\rho^2+2r^2)\cos(\alpha-2\theta) + R^2r\rho^2(-4r^2-3R^2+2\rho^2)\cos(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$+ \frac{\rho^3R^2(2r^2-R^2)\cos(3\alpha-2\theta) + \rho(2R^6+2r^4\rho^2+8r^2R^4-r^2\rho^2R^2-2r^2\rho^4)\cos\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$+ \frac{r(-4R^6-4\rho^2r^2R^2+\rho^4r^2-2\rho^2R^4+2\rho^4R^2)\cos\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$+ \frac{2R^2(R^2-\rho^2)[\rho R^6\cos(\alpha-2\theta) + r^4\rho^3\cos(3\alpha-2\theta) - r^3\rho^2(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$

$$\left. + \frac{2R^4r(R^2-\rho^2)[-R^2(3\rho^2+R^2)\cos\theta + 3\rho r(R^2+\rho^2)\cos\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3} \right\} \tag{A.5}$$

$$\tau_{r\theta}(r,\theta) = \frac{\mu b_y}{\pi(1+\kappa)} \left\{ \frac{r\sin\theta + \rho\sin(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} - \frac{\rho R^2\sin(\alpha-2\theta) + r\rho^2\sin\theta}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} \right.$$

$$+ \frac{r^3\sin\theta + 3r\rho^2\sin(2\alpha-\theta) - 3\rho r^2\sin\alpha - \rho^3\sin(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2}$$

$$- \frac{R^4\rho(R^2-\rho^2)\sin(\alpha-2\theta) + R^2r\rho^2(R^2+2\rho^2)\sin(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$- \frac{-\rho^3R^4\sin(3\alpha-2\theta) + r\rho^2(r^2\rho^2+2R^4-2R^2\rho^2)\sin\theta - 3r^2R^2\rho^3\sin\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$- \frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta) - r^4\rho^3\sin(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$

$$- \frac{2R^4r(R^2-\rho^2)[R^2(3\rho^2+R^2)\sin\theta - 3\rho r(R^2+\rho^2)\sin\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}. \tag{A.6}$$